

## An improper integral and an infinite series

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### A. Baltimore – one of the old cities in the United States

In summer of 2010, I had great time visiting my daughter, Wendy, who is living in a small apartment in Baltimore. She is doing her research work in neuroscience. She came across an improper definite integral when she was reading some papers on quantum mechanics. As I had plenty of time, she gave it to me to kill time.



Wendy, Eliza and I in Inner Harbour, Baltimore

### B. Hold on, the adventure begins here

The integral is: 
$$I = \int_0^{\infty} \frac{x^3}{e^x - 1} dx \quad \dots (1)$$

This integral is interesting as I tried some elementary integration techniques and failed. I doubted that this integral can be solved by employing more advanced methods such as complex analysis using contour integration. However, I really forgot much on the subject and I did not have related books at hands. So I tried something easier that I could handle. I began with an important step to extend (1) to a more general integral :

$$I = \int_0^{\infty} \frac{x^n}{e^x - 1} dx, \text{ where } n \in \mathbf{N} \quad \dots (2)$$

$$= \int_0^{\infty} \frac{x^n e^{-x}}{1 - e^{-x}} dx \quad \dots (3)$$

Using the infinite geometric series :  $\frac{1}{1-r} = 1 + r + r^2 + r^3 + \dots$

We have  $\frac{1}{1 - e^{-x}} = \sum_{k=0}^{\infty} e^{-kx} = 1 + e^{-x} + e^{-2x} + e^{-3x} + \dots$

(3) then becomes:

$$\begin{aligned} I &= \int_0^{\infty} x^n e^{-x} [\sum_{k=0}^{\infty} e^{-kx}] dx = \int_0^{\infty} x^n e^{-x} [1 + e^{-x} + e^{-2x} + e^{-3x} + \dots] dx \\ &= \int_0^{\infty} x^n [e^{-x} + e^{-2x} + e^{-3x} + e^{-4x} \dots] dx \\ &= \int_0^{\infty} x^n [\sum_{k=1}^{\infty} e^{-kx}] dx \quad \dots (4) \end{aligned}$$

Now, let  $u = kx$ ,  $x^n = \frac{u^n}{k^n}$ .  $du = k dx \Rightarrow dx = \frac{du}{k}$

(4) is then transformed into another even more interesting integral :

$$I = \int_0^{\infty} \frac{u^n}{k^n} [\sum_{k=0}^{\infty} e^{-u}] \frac{du}{k} = \sum_{k=0}^{\infty} \frac{1}{k^{n+1}} \int_0^{\infty} u^n e^{-u} du \quad \dots (5)$$

### C. Gamma function

The right-most integral in (5), that is,

$$f(u, n) = \int_0^{\infty} u^n e^{-u} du$$

is the famous Gamma function. It extends the concept of **factorial** to **non-integers**. The study dated back to Leonhard Euler, who gave a formula for its calculation in 1729.



**Leonhard Euler (1707 -1783) in 10 Frank Note, Switzerland**

Leonhard Euler was a Swiss mathematician and physicist. He was one of the greatest mathematicians of the nineteenth century. Euler was the first to use the term "function" to describe an expression involving various arguments; i.e.,  $y = f(x)$ . Also he introduced lasting notation for common geometric functions such as sine, cosine, and tangent. He elaborated the theory of higher transcendental functions by introducing the gamma function and the gamma density functions. He created the Latin square, which likely inspired Howard Garns' number puzzle SuDoku. The asteroid 2002 Euler is named in his honor.

Since  $n$  is an integer here, we just like to show

$$f(u, n) = \int_0^{\infty} u^n e^{-u} du = n!$$

We begin with integration by parts:

$$\begin{aligned} f(u, n) &= \int_0^{\infty} u^n e^{-u} du = - \int_0^{\infty} u^n d(e^{-u}) \\ &= - \left[ u^n e^{-u} \Big|_0^{\infty} - n \int_0^{\infty} u^{n-1} e^{-u} du \right] = - \left[ \lim_{t \rightarrow \infty} t^n e^{-t} - 0 - n \int_0^{\infty} u^{n-1} e^{-u} du \right] \\ &= n \int_0^{\infty} u^{n-1} e^{-u} du \quad \dots \quad (6) \end{aligned}$$

since by L'hospital rule:  $\lim_{t \rightarrow \infty} t^n e^{-t} = \lim_{t \rightarrow \infty} \frac{t^n}{e^t} = \lim_{t \rightarrow \infty} \frac{nt^{n-1}}{e^t} = \lim_{t \rightarrow \infty} \frac{n(n-1)t^{n-2}}{e^t}$

$$= \dots = \lim_{t \rightarrow \infty} \frac{n!}{e^t} = 0$$

By (6), we have:  $f(u, n) = n f(u, n-1) = n(n-1) f(u, n-2) = \dots = n! f(u, 0)$  .... (7)

But  $f(u, 0) = \int_0^{\infty} u^0 e^{-u} du = \int_0^{\infty} e^{-u} du = -e^{-u} \Big|_0^{\infty} = 1$ .

Therefore  $f(u, n) = \int_0^{\infty} u^n e^{-u} du = n! f(u, 0) = n!$  .... (8)

## D. Reimann Zeta function

Substitute (8) in (5), we get:  $I = \int_0^{\infty} \frac{x^n}{e^x - 1} dx = \sum_{k=0}^{\infty} \frac{1}{k^{n+1}} n!$  .... (9)

Back from the future! We go back to (1), and take  $n$  to be 3.

(9) becomes:  $I = \int_0^{\infty} \frac{x^3}{e^x - 1} dx = \sum_{k=0}^{\infty} \frac{1}{k^{3+1}} 3! = 6 \sum_{k=0}^{\infty} \frac{1}{k^4}$  .... (10)

The function  $Z(n) = \sum_{k=1}^{\infty} \frac{1}{k^n} = 1 + \frac{1}{2^n} + \frac{1}{3^n} + \frac{1}{4^n} + \dots$  is called the Riemann Zeta function.

For  $n = 1$ , the function is called the harmonic series and is divergent.

For  $n > 1$ , the zeta function converges.

Bernhard Riemann (1826-1866) was one of the leading mathematicians of the nineteenth century. In his short career, he introduced ideas of fundamental importance in complex analysis, real analysis, differential geometry, number theory, and other subjects. His work in differential geometry provided the mathematical basis for the general theory of relativity.



**Bernhard Riemann (1826-1866)**

From (10), we move our story to find the infinite series:

$$Z(4) = \sum_{k=1}^{\infty} \frac{1}{k^4} = 1 + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \dots$$

Since  $Z(4)$  is rather complicated in evaluation, we like to begin the proof of a more historical one:

$$Z(2) = \sum_{k=1}^{\infty} \frac{1}{k^2} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \dots = \frac{\pi^2}{6} \quad \text{(Basel problem)}$$

$Z(2)$  is an interesting and mysterious function and many mathematicians in the nineteenth century have great works on this subject. It is mysterious because the result involves  $\pi$ .

There are many different methods in evaluating  $Z(2)$ , most of them involves advanced mathematics. I like to choose a relatively easier method, the powerful Fourier series.

## E. Fourier series

Fourier was a French mathematician, who was taught by Lagrange and Laplace. He almost died on the guillotine in the French Revolution. Fourier was a buddy of Napoleon and worked as scientific adviser for Napoleon's army. He worked on theories of heat and expansions of functions as trigonometric series. All these subjects were controversial at the time. Like many scientists, he had to battle to get his ideas accepted.



**Jean Baptiste Joseph Fourier (1768 - 1830)**

Let us start with :

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \quad \dots \quad (11)$$

, where  $a_i, b_i$  are constants.

The series is periodic with period  $2\pi$  and we take  $-\pi < x < \pi$ .

To find  $a_n$  we multiply (11) by  $\cos nx$  and then integrate from  $-\pi$  to  $\pi$ .

$$\int_{-\pi}^{\pi} f(x) \cos nx \, dx = \pi a_n \quad \dots \quad (12)$$

$$\begin{aligned} \text{(a) For } m \neq n, \quad \int_{-\pi}^{\pi} \cos mx \cos nx \, dx &= \frac{1}{2} \int_{-\pi}^{\pi} [\cos(m+n)x + \cos(m-n)x] \, dx \\ &= \frac{1}{2} \left[ \frac{\sin(m+n)x}{m+n} + \frac{\sin(m-n)x}{m-n} \right]_{-\pi}^{\pi} = 0 \end{aligned}$$

$$\text{(b) For } m = n, \quad \int_{-\pi}^{\pi} \cos^2 nx \, dx = \frac{1}{2} \int_{-\pi}^{\pi} (1 + \cos 2nx) \, dx = \left[ \frac{1}{2}x + \frac{\sin 2nx}{4n} \right]_{-\pi}^{\pi} = \pi$$

$$\text{(c) For all } m, \quad \int_{-\pi}^{\pi} \cos mx \sin nx \, dx = 0$$

$$\therefore a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx \, dx \quad \dots \quad (13)$$

$$\text{Similarly, } b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx \, dx \quad \dots \quad (14)$$

Note that :

$$\text{(a) } a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \, dx \quad \dots \quad (15)$$

In order that this is compatible with (13), we use  $\frac{a_0}{2}$  instead of  $a_0$  in (11).

(b) There is no  $b_0$ .

## F. The evaluation of Zeta $Z(2)$

Consider the Fourier expansion of  $f(x) = x^2$ .

Since  $x^2 \sin nx$  is an odd function with period  $2\pi$ ,  $b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} x^2 \sin nx \, dx = 0$

$$\text{By (15), } a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} x^2 \, dx = \frac{1}{\pi} \left[ \frac{x^3}{3} \right]_{-\pi}^{\pi} = \frac{2\pi^2}{3}$$

$$\begin{aligned} \text{By (13), } a_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} x^2 \cos nx \, dx = \frac{1}{n\pi} \int_{-\pi}^{\pi} x^2 \, d(\sin nx) \\ &= \frac{1}{n\pi} \{ [x^2 (\sin nx)]_{-\pi}^{\pi} - \int_{-\pi}^{\pi} 2x (\sin nx) \, dx \} \end{aligned}$$

$$\begin{aligned}
&= -\frac{2}{n\pi} \int_{-\pi}^{\pi} x (\sin nx) dx = \frac{2}{n^2\pi} \int_{-\pi}^{\pi} x d(\cos nx) \\
&= \frac{2}{n^2\pi} \{ [x (\cos nx)]_{-\pi}^{\pi} - \int_{-\pi}^{\pi} \cos nx dx \} = \frac{2}{n^2\pi} \{ [2\pi(-1)^n] - \frac{1}{n} [\sin nx]_{-\pi}^{\pi} \} \\
&= \frac{4(-1)^n}{n^2}
\end{aligned}$$

Substitute the values of  $a_0, a_n, b_n$  in (11),  $x^2 = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \left( \frac{4(-1)^n}{n^2} \cos nx \right)$

$$\text{Now, put } x = \pi, \quad \pi^2 = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \left( \frac{4(-1)^n}{n^2} \cos n\pi \right) = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \left( \frac{4(-1)^n}{n^2} (-1)^n \right)$$

$$\therefore \pi^2 = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \left( \frac{4}{n^2} \right)$$

$$\therefore Z(2) = \sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \dots = \frac{\pi^2}{6} \quad \dots \quad (16)$$

#### F. The infinite series Zeta $Z(4)$ and the original integral

The evaluation of  $Z(4)$  is basically the same as  $Z(2)$ , but is more complicated in the workings. Consider the Fourier expansion of  $f(x) = x^4$ .

Since  $x^4 \sin nx$  is an odd function,  $b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} x^4 \sin nx dx = 0$

$$\text{By (15), } a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} x^4 dx = \frac{1}{\pi} \left[ \frac{x^5}{5} \right]_{-\pi}^{\pi} = \frac{2\pi^4}{5}$$

$$\begin{aligned}
\text{By (13), } a_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} x^4 \cos nx dx = \frac{1}{n\pi} \int_{-\pi}^{\pi} x^4 d(\sin nx) \\
&= \frac{1}{n\pi} \{ [x^4 (\sin nx)]_{-\pi}^{\pi} - \int_{-\pi}^{\pi} 4x^3 (\sin nx) dx \} \\
&= -\frac{4}{n\pi} \int_{-\pi}^{\pi} x^3 (\sin nx) dx = \frac{4}{n^2\pi} \int_{-\pi}^{\pi} x^3 d(\cos nx) \\
&= \frac{4}{n^2\pi} \{ [x^3 (\cos nx)]_{-\pi}^{\pi} - \int_{-\pi}^{\pi} 3x^2 \cos nx dx \} \\
&= \frac{4}{n^2\pi} \{ [2\pi^3(-1)^n] - \frac{3}{n} \int_{-\pi}^{\pi} x^2 d(\sin nx) \} \\
&= \frac{4}{n^2\pi} \{ [2\pi^3(-1)^n] - \frac{3}{n} [x^2 (\sin nx)]_{-\pi}^{\pi} - \frac{3}{n} \int_{-\pi}^{\pi} 2x (\sin nx) dx \} \\
&= \frac{4}{n^2\pi} \{ [2\pi^3(-1)^n] - \frac{6}{n^2} \int_{-\pi}^{\pi} x d(\cos nx) \} \\
&= \frac{8\pi^2(-1)^n}{n^2} - \frac{24}{n^4\pi} \{ [x (\cos nx)]_{-\pi}^{\pi} - \int_{-\pi}^{\pi} \cos nx dx \}
\end{aligned}$$

$$\begin{aligned}
&= \frac{8\pi^2(-1)^n}{n^2} - \frac{24}{n^4\pi} \{ [x (\cos nx)]_{-\pi}^{\pi} - \int_{-\pi}^{\pi} \cos nx \, dx \} \\
&= \frac{8\pi^2(-1)^n}{n^2} - \frac{24}{n^4\pi} \{ [2\pi(-1)^n] - \frac{1}{n} [\sin nx]_{-\pi}^{\pi} \} \\
&= \frac{8\pi^2(-1)^n}{n^2} - \frac{48(-1)^n}{n^4}
\end{aligned}$$

Substitute the values of  $a_0, a_n, b_n$  in (11),

$$x^4 = \frac{\pi^4}{5} + \sum_{n=1}^{\infty} \left[ \left( \frac{8\pi^2(-1)^n}{n^2} - \frac{48(-1)^n}{n^4} \right) \cos nx \right]$$

Now, put  $x = \pi$ ,  $\pi^4 = \frac{\pi^4}{5} + \sum_{n=1}^{\infty} \left[ \left( \frac{8\pi^2(-1)^n}{n^2} - \frac{48(-1)^n}{n^4} \right) (-1)^n \right]$

$$\therefore \pi^4 = \frac{\pi^4}{5} + 8\pi^2 \sum_{n=1}^{\infty} \left( \frac{1}{n^2} \right) - 48 \sum_{n=1}^{\infty} \left( \frac{1}{n^4} \right)$$

$$\pi^4 = \frac{\pi^4}{5} + 8\pi^2 \left( \frac{\pi^2}{6} \right) - 48 \sum_{n=1}^{\infty} \left( \frac{1}{n^4} \right) \quad , \text{ by (15)}$$

$$\therefore Z(4) = \sum_{n=1}^{\infty} \frac{1}{n^4} = 1 + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \dots = \frac{\pi^4}{90} \quad \dots \quad (17)$$

Finally we finish our long way journey. From (10), we have

$$I = \int_0^{\infty} \frac{x^3}{e^x - 1} dx = 6 \sum_{k=0}^{\infty} \frac{1}{k^4} = 6 \left( \frac{\pi^4}{90} \right) = \frac{\pi^4}{15}$$